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Regular Partitions of Hypergraphs: Counting Lemmas

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We continue the study of regular partitions of hypergraphs. In particular, we obtain corresponding *counting lemmas* for the *regularity lemmas* for hypergraphs from our paper 'Regular Partitions of Hypergraphs: Regularity Lemmas' (in this issue).

1. Introduction

In this paper we continue the line of research from [4, 8, 11, 13] and obtain the corresponding counting lemmas, Theorem 1.2 and Theorem 1.3, for the regularity lemmas from [11]. A standard application of those theorems, following the lines of [3, 4, 5, 8, 14], yields a proof of the so-called removal lemma for hypergraphs. Moreover, those new lemmas have already been used for other applications in [1, 2, 9, 10, 12].

1.1. Basic notation

For real constants α, β, and a non-negative constant ξ we sometimes write

$$\alpha = \beta \pm \xi, \quad \text{if } \beta - \xi \leqslant \alpha \leqslant \beta + \xi.$$

For a positive integer ℓ, we denote by $[\ell]$ the set $\{1,\dots,\ell\}$. For a set V and an integer $k \geqslant 1$, let $[V]^k$ be the set of all k-element subsets of V. We may drop one pair of brackets and write $[\ell]^k$ instead of $\big[[\ell]\big]^k$. A subset $\mathcal{H}^{(k)} \subseteq [V]^k$ is a *k-uniform hypergraph* on the vertex set V. We identify hypergraphs with their edge sets. For a given k-uniform hypergraph $\mathcal{H}^{(k)}$, we denote by $V(\mathcal{H}^{(k)})$ and $E(\mathcal{H}^{(k)})$ its vertex and edge set, respectively. For $U \subseteq V(\mathcal{H}^{(k)})$, we denote by $\mathcal{H}^{(k)}[U]$ the sub-hypergraph of $\mathcal{H}^{(k)}$ induced on U (*i.e.*, $\mathcal{H}^{(k)}[U] = \mathcal{H}^{(k)} \cap [U]^k$). A k-uniform

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clique of *order* j, denoted by $K_j^{(k)}$, is a k-uniform hypergraph on $j \geqslant k$ vertices consisting of all $\binom{j}{k}$ different k-tuples.

In this paper, ℓ-partite, j-uniform hypergraphs play a special rôle, where $j \leqslant \ell$. Given vertex sets $V_1, \ldots, V_\ell$, we denote by $K_\ell^{(j)}(V_1, \ldots, V_\ell)$ the *complete* ℓ-partite, j-uniform hypergraph (*i.e.*, the family of all j-element subsets $J \subseteq \bigcup_{i \in [\ell]} V_i$ satisfying $|V_i \cap J| \leqslant 1$ for every $i \in [\ell]$). If $|V_i| = m$ for every $i \in [\ell]$, then an *(m, ℓ, j)-hypergraph* $\mathcal{H}^{(j)}$ on $V_1 \cup \cdots \cup V_\ell$ is any subset of $K_\ell^{(j)}(V_1, \ldots, V_\ell)$. Note that the vertex partition $V_1 \cup \cdots \cup V_\ell$ is an $(m, \ell, 1)$-hypergraph $\mathcal{H}^{(1)}$. (This definition may seem artificial right now, but it will simplify later notation.) For $j \leqslant i \leqslant \ell$ and set $\Lambda_i \in [\ell]^i$, we denote by $\mathcal{H}^{(j)}[\Lambda_i] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda \in \Lambda_i} V_\lambda\big]$ the sub-hypergraph of the (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ induced on $\bigcup_{\lambda \in \Lambda_i} V_\lambda$.

For an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ and an integer $j \leqslant i \leqslant \ell$, we denote by $\mathcal{K}_i(\mathcal{H}^{(j)})$ the family of all i-element subsets of $V(\mathcal{H}^{(j)})$ which span complete sub-hypergraphs in $\mathcal{H}^{(j)}$ of order i. Note that $|\mathcal{K}_i(\mathcal{H}^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $\mathcal{H}^{(j)}$.

Given an $(m, \ell, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ and an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ such that $V(\mathcal{H}^{(j)}) \subseteq V(\mathcal{H}^{(j-1)})$, we say an edge J of $\mathcal{H}^{(j)}$ *belongs to* $\mathcal{H}^{(j-1)}$ if $J \in \mathcal{K}_j(\mathcal{H}^{(j-1)})$, *i.e.*, J corresponds to a clique of order j in $\mathcal{H}^{(j-1)}$. Moreover, $\mathcal{H}^{(j-1)}$ *underlies* $\mathcal{H}^{(j)}$ if $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$, *i.e.*, every edge of $\mathcal{H}^{(j)}$ belongs to $\mathcal{H}^{(j-1)}$. This brings us to one of the main concepts of this paper, the notion of a *complex*.

Definition 1 ((m, ℓ, h)-complex). Let $m \geqslant 1$ and $\ell \geqslant h \geqslant 1$ be integers. An *(m, ℓ, h)-complex* $\boldsymbol{\mathcal{H}}$ is a collection of (m, ℓ, j)-hypergraphs $\{\mathcal{H}^{(j)}\}_{j=1}^h$ such that

(a) $\mathcal{H}^{(1)}$ is an $(m, \ell, 1)$-hypergraph, *i.e.*, $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$,
(b) $\mathcal{H}^{(j-1)}$ underlies $\mathcal{H}^{(j)}$ for $2 \leqslant j \leqslant h$, *i.e.*, $\mathcal{H}^{(j)} \subseteq \mathcal{K}_j(\mathcal{H}^{(j-1)})$.

1.2. Regular complexes

We begin with a notion of relative density of a j-uniform hypergraph w.r.t. a $(j-1)$-uniform hypergraph on the same vertex set.

Definition 2 (relative density). Let $\mathcal{H}^{(j)}$ be a j-uniform hypergraph and let $\mathcal{H}^{(j-1)}$ be a $(j-1)$-uniform hypergraph on the same vertex set. We define the *density of* $\mathcal{H}^{(j)}$ *w.r.t.* $\mathcal{H}^{(j-1)}$ as

$$d\big(\mathcal{H}^{(j)} \mid \mathcal{H}^{(j-1)}\big) = \begin{cases} \frac{|\mathcal{H}^{(j)} \cap \mathcal{K}_j(\mathcal{H}^{(j-1)})|}{|\mathcal{K}_j(\mathcal{H}^{(j-1)})|} & \text{if } |\mathcal{K}_j(\mathcal{H}^{(j-1)})| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

We now define a notion of regularity of an (m, j, j)-hypergraph with respect to an $(m, j, j-1)$-hypergraph.

Definition 3. Let reals $\varepsilon > 0$ and $d_j \geqslant 0$ be given along with an (m, j, j)-hypergraph $\mathcal{H}^{(j)}$ and an underlying $(m, j, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$. We say $\mathcal{H}^{(j)}$ is *(ε, d_j)-regular* w.r.t. $\mathcal{H}^{(j-1)}$ if, whenever $\mathcal{Q}^{(j-1)} \subseteq \mathcal{H}^{(j-1)}$ satisfies

$$|\mathcal{K}_j(\mathcal{Q}^{(j-1)})| \geqslant \varepsilon |\mathcal{K}_j(\mathcal{H}^{(j-1)})|, \quad \text{then} \quad d\big(\mathcal{H}^{(j)} \mid \mathcal{Q}^{(j-1)}\big) = d_j \pm \varepsilon.$$

Next we extend the notion of (ε, d_j)-regularity from (m, j, j)-hypergraphs to (m, ℓ, j)-hypergraphs $\mathcal{H}^{(j)}$.

Definition 4 (**(ε, d_j)-regular hypergraph).** We say an (m, ℓ, j)-hypergraph $\mathcal{H}^{(j)}$ is (ε, d_j)-*regular* w.r.t. an $(m, \ell, j-1)$-hypergraph $\mathcal{H}^{(j-1)}$ if for every $\Lambda_j \in [\ell]^j$ the restriction $\mathcal{H}^{(j)}[\Lambda_j] = \mathcal{H}^{(j)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$ is (ε, d_j)-regular w.r.t. to the restriction $\mathcal{H}^{(j-1)}[\Lambda_j] = \mathcal{H}^{(j-1)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$.

We sometimes write ε-*regular* to mean $\big(\varepsilon, d\big(\mathcal{H}^{(j)} \mid \mathcal{H}^{(j-1)}\big)\big)$-regular.

Finally, we close this section with the notion of a regular complex.

Definition 5 (**$(\varepsilon, \boldsymbol{d})$-regular complex).** Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \ldots, d_h)$ be a vector of non-negative reals. We say an (m, ℓ, h)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is $(\varepsilon, \boldsymbol{d})$-*regular* if $\mathcal{H}^{(j)}$ is (ε, d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}$ for every $j = 2, \ldots, h$.

1.3. Main results

In this paper we prove the counting lemmas corresponding to the regularity lemmas from [11]. Such a counting lemma should ensure the 'right' number of copies of a given k-uniform hypergraph in an appropriate collection of dense and regular polyads provided by the corresponding regularity lemma. Here the 'right' number means that the number of copies is approximately the same as in the random object of the same density. For example, the following well-known fact is the counting lemma corresponding to Szemerédi's regularity lemma for graphs, restricted to the case of estimating the number of cliques.

Fact 1.1 (counting lemma). *For every integer ℓ and positive reals d and γ there exists $\varepsilon > 0$ so that the following holds. Let $G = \bigcup_{1\leqslant i<j\leqslant \ell} G^{ij}$ be an ℓ-partite graph with ℓ-partition $V_1 \cup \cdots \cup V_\ell$, where $G^{ij} = G[V_i, V_j]$, $1 \leqslant i < j \leqslant \ell$, and $|V_1| = \cdots = |V_\ell| = n$. Suppose further that all graphs G^{ij} are ε-regular with density d. Then the number of copies of the ℓ-clique K_ℓ in G is within the interval $(1 \pm \gamma)d^{\binom{\ell}{2}}n^\ell$.*

In order to avoid some technical details, for the hypergraph case we restrict our attention to the lower bound only. We first state the counting lemma for [11, Theorem 2.13], for which we use the following notation.

Definition 6 (**ν-close).** Let m and $\ell \geqslant k \geqslant 2$ be integers and $\nu > 0$, let $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ be an $(m, \ell, k-1)$-complex, and let $\mathcal{H}^{(k)}$ and $\mathcal{G}^{(k)}$ be k-uniform sub-hypergraphs of $\mathcal{K}_k(\mathcal{R}^{(k-1)})$. We say $\mathcal{H}^{(k)}$ and $\mathcal{G}^{(k)}$ are ν-*close* w.r.t. $\boldsymbol{\mathcal{R}}$ if, for every $\Lambda_k \in [\ell]^k$, we have

$$\Big|\Big(\mathcal{H}^{(k)} \cap \mathcal{K}_k\big(\mathcal{R}^{(k-1)}[\Lambda_k]\big)\Big) \triangle \Big(\mathcal{G}^{(k)} \cap \mathcal{K}_k\big(\mathcal{R}^{(k-1)}[\Lambda_k]\big)\Big)\Big| \leqslant \nu|\mathcal{K}_k(\mathcal{R}^{(k-1)})|.$$

The following lemma estimates the number of cliques in a hypergraph $\mathcal{H}^{(k)}$, which is ν-close to an ε-regular hypergraph $\mathcal{G}^{(k)}$.

Theorem 1.2. *For all integers $\ell \geqslant k \geqslant 2$ and all constants $\gamma > 0$ and $d_k > 0$, there is some $\nu > 0$ such that, for every $d_0 > 0$, there exist $\varepsilon > 0$ and m_0 such that the following holds.*

Suppose

(i) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ *is an* $(\varepsilon, (d_2, \ldots, d_{k-1}))$*-regular* $(m, \ell, k-1)$*-complex with* $d_i \geqslant d_0$ *for every* $i = 2, \ldots, k-1$ *and* $m \geqslant m_0$*,*
(ii) $\mathcal{G}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ *is* (ε, d_k)*-regular w.r.t.* $\mathcal{R}^{(k-1)}[\Lambda_k]$ *for every* $\Lambda_k \in [\ell]^k$*, and*
(iii) $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ *is* ν*-close to* $\mathcal{G}^{(k)}$ *w.r.t.* $\boldsymbol{\mathcal{R}}$*.*

Then

$$|\mathcal{K}_\ell(\mathcal{H}^{(k)})| \geqslant (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell.$$

We give the details of the proof of Theorem 1.2 in Section 3. Basically, it will follow from the 'closeness' of $\mathcal{H}^{(k)}$ and $\mathcal{G}^{(k)}$ (*cf.* (iii)) that the number of $\mathcal{K}_\ell^{(k)}$s in $\mathcal{G}^{(k)} \cap \mathcal{H}^{(k)}$ will be essentially the same as in $\mathcal{G}^{(k)}$. Therefore, in order to prove Theorem 1.2 it suffices to find a lower bound on the number of such cliques in $\mathcal{G}^{(k)}$. For that we will make use of the so-called *dense counting lemma* (see Theorem 2.1 below) which was proved by Kohayakawa, Rödl and Skokan [6]. The dense counting lemma estimates the number of $\mathcal{K}_\ell^{(k)}$s in a 'densely regular' complex such as $\{\mathcal{R}^{(1)}, \ldots, \mathcal{R}^{(k-1)}, \mathcal{G}^{(k)}\}$. Here 'densely regular' means that the measure of regularity is much smaller then the densities of the complex in which one wants to count, *i.e.*, $\varepsilon \ll d_i$ for all $i = 2, \ldots, k$. In other words, compared to the measure of regularity the complex is relatively dense in every layer.

Note that such a 'densely regular' environment cannot be enforced by an application of the regularity lemma, since δ_k is independent of $a_2, \ldots, a_{k-1}$. Consequently, a counting lemma useful in conjunction with [11, Theorem 2.16] has to allow the following hierarchy of the constants:

$$d_k \gg \delta_k \gg d_{k-1} = a_{k-1}^{-1},\, d_{k-2} = a_{k-2}^{-1}, \ldots, d_2 = a_2^{-1} \geqslant \delta, \frac{1}{r}. \tag{1.1}$$

The methods developed in this paper allow a simple proof of the following theorem, which matches the hierarchy in (1.1).

Theorem 1.3. *For all integers* $\ell \geqslant k \geqslant 2$ *and positive constants* $\gamma > 0$ *and* $d_k > 0$*, there exist* $\delta_k > 0$ *such that, for every* $d_{k-1}, \ldots, d_2 > 0$ *with* $\frac{1}{d_i} \in \mathbb{N}$*, for every* $i = 2, \ldots, k-1$*, there are constants* $\delta > 0$ *and positive integers* r *and* m_0 *so that the following holds.*

Suppose

(i) $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ *is an* $(\delta, (d_2, \ldots, d_{k-1}))$*-regular* $(m, \ell, k-1)$*-complex with* $m \geqslant m_0$*, and*
(ii) $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ *is* (δ_k, d_k, r)*-regular w.r.t.* $\mathcal{R}^{(k-1)}[\Lambda_k]$ *for every* $\Lambda_k \in [\ell]^k$*.*

Then

$$|\mathcal{K}_\ell(\mathcal{H}^{(k)})| \geqslant (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^\ell.$$

We note that the condition that $\frac{1}{d_i} \in \mathbb{N}$ for $i = 2, \ldots, k-1$ in (i) is not restrictive. This is because the hypergraph regularity lemma provides a partition $\mathscr{P}$ in which all densities of the underlying structure satisfy this condition (*i.e.*, $d_i = \frac{1}{a_i}$ for $i = 2, \ldots, k-1$).

2. The dense counting and extension lemma

The proof of Theorem 1.2 and Theorem 1.3 relies on the so-called *dense counting lemma* from [6] This theorem can be used to estimate the number of copies of $K_\ell^{(h)}$ in an appropriate collection of dense and regular blocks within a regular partition provided by the regular approximation lemma [11, Theorem 2.13]. Moreover, it can be applied to count the number of $K_k^{(k-1)}$s in the polyads of the partitions obtained by the regularity lemmas from [11].

Theorem 2.1 (dense counting lemma). *For all integers $2 \leqslant h \leqslant \ell$ and all positive constants γ and d_0 there exist $\varepsilon_{\mathrm{DCL}} = \varepsilon_{\mathrm{DCL}}(h,\ell,\gamma,d_0) > 0$ and an integer $m_{\mathrm{DCL}} = m_{\mathrm{DCL}}(h,\ell,\gamma,d_0)$ so that if $\boldsymbol{d} = (d_2,\ldots,d_h) \in \mathbb{R}^{h-1}$ satisfying $d_j \geqslant d_0$ for $2 \leqslant j \leqslant h$ and $m \geqslant m_{\mathrm{DCL}}$, and if $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is an $(\varepsilon_{\mathrm{DCL}}, \boldsymbol{d})$-regular (m,ℓ,h)-complex, then*

$$|\mathcal{K}_\ell(\mathcal{H}^{(h)})| = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{\ell}{j}} \times m^\ell.$$

This theorem was proved by Kohayakawa, Rödl and Skokan in [6, Theorem 6.5]. The proof presented there was based on a double induction over the uniformity h and the number of vertices of $\mathcal{F}^{(h)}$. As it turned out, a double induction over h and the number of edges in $\mathcal{F}^{(h)}$ allows a somewhat simpler argument and we will follow this idea. In that sense the proof presented here is similar to the proof of the counting lemma in [15]. Due to the induction we prove a slightly more general statement (see Theorem 2.2 below). The generalization of Theorem 2.1 allows us to estimate the number of copies of an arbitrary hypergraph $\mathcal{F}^{(h)}$ with vertices $\{1,\ldots,\ell\}$ in an (m,ℓ,k)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ satisfying that $\mathcal{H}^{(j)}[\Lambda_j]$ is regular w.r.t. $\mathcal{H}^{(j-1)}[\Lambda_j]$ whenever $\Lambda_j \subseteq e$ for some edge e of $\mathcal{F}^{(h)}$. Rather than counting copies of K_ℓ in an 'everywhere' regular complex, this lemma counts copies of $\mathcal{F}^{(h)}$ in $\mathcal{H}^{(h)}$ satisfying the less restrictive assumptions above. We introduce some more notation before we give the precise statement below (see Theorem 2.2).

For a fixed h-uniform hypergraph $\mathcal{F}^{(h)}$, we define the jth shadow for $j \in [h]$ by

$$\Delta_j(\mathcal{F}^{(h)}) = \{J : |J| = j \text{ and } J \subseteq f \text{ for some edge } f \in \mathcal{F}^{(h)}\}.$$

We extend the notion of an $(\varepsilon, \boldsymbol{d})$-regular complex (*cf.* Definition 5) to an $(\varepsilon, \boldsymbol{d}, \boldsymbol{\mathcal{F}})$-regular complex.

Definition 7 ($(\varepsilon, \boldsymbol{d}, \boldsymbol{\mathcal{F}})$-regular complex). Let ε be a positive real and let $\boldsymbol{d} = (d_2,\ldots,d_h)$ be a vector of non-negative reals. Let $\boldsymbol{\mathcal{F}} = \{\mathcal{F}^{(j)}\}_{j=1}^h$ be a $(1,\ell,h)$-complex on ℓ vertices $\{1,\ldots,\ell\}$. We say an (m,ℓ,h)-complex $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ with vertex partition $\mathcal{H}^{(1)} = V_1 \cup \cdots \cup V_\ell$ is $(\varepsilon, \boldsymbol{d}, \boldsymbol{\mathcal{F}})$*-regular* if, for every $2 \leqslant j \leqslant h$, the following holds:

(a) for all $\Lambda_j \in \mathcal{F}^{(j)}$ the (m,j,j)-hypergraph $\mathcal{H}^{(j)}[\Lambda_j]$ is (ε, d_j)-regular w.r.t. $\mathcal{H}^{(j-1)}[\Lambda_j]$, and
(b) for all $\Lambda_j \notin \mathcal{F}^{(j)}$ the (m,j,j)-hypergraph $\mathcal{H}^{(j)}[\Lambda_j]$ is empty.

Definition 7 imposes a regular structure on those (m,j,j)-subcomplexes of $\mathcal{H}^{(j)}$ which naturally correspond to edges of the hypergraph $\mathcal{F}^{(j)}$ (*i.e.*, on a subcomplex induced on $V_{\lambda_1},\ldots,V_{\lambda_j}$, where $\{\lambda_1,\ldots,\lambda_j\}$ forms an edge in $\mathcal{F}^{(j)}$). We need one more definition before we can state the generalization of Theorem 2.1.

Definition 8 (partite isomorphic). Let $\mathcal{F} = \{\mathcal{F}^{(j)}\}_{j=1}^h$ be a $(1, \ell, h)$-complex with $V(\mathcal{F}^{(1)}) = [\ell]$ and $\mathcal{H} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ an (m, ℓ, h)-complex with vertex partition $V(\mathcal{H}^{(1)}) = V_1 \cup \cdots \cup V_\ell$. We say a copy $\mathcal{F}_0$ of $\mathcal{F}$ in $\mathcal{H}$ is *partite isomorphic* to $\mathcal{F}$ if there is a labelling of $V(\mathcal{F}_0^{(1)}) = \{v_1, \ldots, v_\ell\}$ such that

(i) $v_i \in V_i$ for every $i \in [\ell]$, and
(ii) $v_i \mapsto i$ is a hypergraph isomorphism (edge-preserving bijection of the vertex sets) between $\mathcal{F}_0^{(j)}$ and $\mathcal{F}^{(j)}$ for every $j = 1, \ldots, h$.

The following theorem is a generalization of Theorem 2.1.

Theorem 2.2 (general dense counting lemma). *For all integers $1 \leqslant h \leqslant \ell$, every $(1, \ell, h)$-complex $\mathcal{F} = \{\mathcal{F}^{(j)}\}_{j=1}^h$, and all positive constants γ and d_0, there exist $\varepsilon = \varepsilon(\mathcal{F}, \gamma, d_0) > 0$ and an integer $m_0 = m_0(\mathcal{F}, \gamma, d_0)$ such that if $\boldsymbol{d} = (d_2, \ldots, d_h) \in \mathbb{R}^{h-1}$ satisfies $d_j \geqslant d_0$ for $2 \leqslant j \leqslant h$ and $m \geqslant m_0$, and if $\mathcal{H} = \{\mathcal{H}^{(j)}\}_{j=1}^h$ is an $(\varepsilon, \boldsymbol{d}, \mathcal{F})$-regular (m, ℓ, h)-complex, then the number of partite isomorphic copies of $\mathcal{F}$ in $\mathcal{H}$ is*

$$(1 \pm \gamma) \prod_{j=2}^{h} d_j^{|\mathcal{F}^{(j)}|} \times m^\ell.$$

Proof. Theorem 2.2 is trivial if $h = 1$. (Alternatively, we could start the induction with $h = 2$, for which Theorem 2.2 reduces to the well-known counting lemma for graphs (see, *e.g.*, [7])).

Let $h \geqslant 2$. If $\mathcal{F}^{(h)} = \varnothing$, then Theorem 2.2 follows from the induction assumption for $h - 1$. So let $|\mathcal{F}^{(h)}| \geqslant 1$ and positive constants γ and d_0 be given. Fix some arbitrary edge $e \in \mathcal{F}^{(h)}$ and let $\mathcal{F}_-^{(h)} = \mathcal{F}^{(h)} \setminus e$ and $\mathcal{F}_- = \{\mathcal{F}^{(1)}, \ldots, \mathcal{F}^{(h-1)}, \mathcal{F}_-^{(h)}\}$. We set

$$\varepsilon = \min\Big\{\varepsilon_{\text{Thm 2.2}}(\mathcal{F}_-, \gamma/2, d_0), \tfrac{\gamma}{2} d_0^{\sum_{j=2}^h |\mathcal{F}^{(j)}|}\Big\},$$

and let m_0 be sufficiently large.

Let $\mathcal{H}$ be an $(\varepsilon, \boldsymbol{d}, \mathcal{F})$-regular (m, ℓ, h)-complex. Set $\mathcal{H}_-^{(h)} = \mathcal{H}^{(h)} \setminus \mathcal{H}^{(h)}[e]$, *i.e.*, we obtain $\mathcal{H}_-^{(h)}$ from $\mathcal{H}^{(h)}$ by removing those edges which are spanned by the vertex classes $V_{i_1} \cup \cdots \cup V_{i_h}$ indexed by elements of $e = \{i_1, \ldots, i_h\} \in [\ell]^h$. Moreover, let $\mathcal{H}_- = \{\mathcal{H}^{(1)}, \ldots, \mathcal{H}^{(h-1)}, \mathcal{H}_-^{(h)}\}$. Clearly, $\mathcal{H}_-$ is an $(\varepsilon, \boldsymbol{d}, \mathcal{F}_-)$-regular (m, ℓ, h)-complex and, due to the choice of ε and the induction assumption on the number edges in $\mathcal{F}_-^{(h)}$, the number $\#\{\mathcal{F}_- \subseteq \mathcal{H}_-\}$ of partite isomorphic copies of $\mathcal{F}_-$ in $\mathcal{H}_-$ is

$$\#\{\mathcal{F}_- \subseteq \mathcal{H}_-\} = \left(1 \pm \frac{\gamma}{2}\right) \prod_{j=2}^{h-1} d_j^{|\mathcal{F}^{(j)}|} \times d_h^{|\mathcal{F}^{(h)}|-1} \times m^\ell. \tag{2.1}$$

For a partite isomorphic copy $\mathcal{F}_{-,0} = \{\mathcal{F}_0^{(1)}, \ldots, \mathcal{F}_0^{(h-1)}, \mathcal{F}_{-,0}^{(h)}\}$ of $\mathcal{F}$ in $\mathcal{H}$, let $\eta(\mathcal{F}_{-,0})$ be the unique set of those h vertices for which $\{\mathcal{F}_0^{(1)}, \ldots, \mathcal{F}_0^{(h-1)}, \mathcal{F}_{-,0}^{(h)} \cup \eta(\mathcal{F}_{-,0})\}$ is a partite isomorphic copy of $\mathcal{F}$. Note that $\eta(\mathcal{F}_{-,0})$ does not necessarily span an edge in $\mathcal{H}^{(h)}$. We denote by $1_{\mathcal{H}^{(h)}}(\eta(\mathcal{F}_{-,0}))\colon \mathcal{H}^{(h)} \to \{0, 1\}$ the indicator function, indicating if the edge is present or not, *i.e.*, $1_{\mathcal{H}^{(h)}}(\eta(\mathcal{F}_{-,0})) = 1$ if and only if $\eta(\mathcal{F}_{-,0}) \in \mathcal{H}^{(h)}$. Hence, the number $\#\{\mathcal{F} \subseteq \mathcal{H}\}$ of partite isomorphic copy of $\mathcal{F}$ in $\mathcal{H}$ equals

$$\begin{aligned}\#\{\mathcal{F}\subseteq\mathcal{H}\} &= \sum\{1_{\mathcal{H}^{(h)}}\big(\eta(\mathcal{F}_{-,0})\big):\ \mathcal{F}_{-,0}\ \text{ is a partite isomorphic copy of } \mathcal{F}_-\ \text{in } \mathcal{H}_-\}\\ &= \sum_{\mathcal{F}_{-,0}}(d_h + 1_{\mathcal{H}^{(h)}}\big(\eta(\mathcal{F}_{-,0})\big) - d_h)\\ &= \#\{\mathcal{F}_-\subseteq\mathcal{H}_-\}\times d_h \pm \left|\sum_{\mathcal{F}_{-,0}} 1_{\mathcal{H}^{(h)}}\big(\eta(\mathcal{F}_{-,0})\big) - d_h\right|. \end{aligned} \tag{2.2}$$

Due to (2.1) we have good control of the first term in (2.2) and we will bound the contribution of the '$\pm$-term' using the regularity of $\mathcal{H}$. For that, consider the induced sub-complexes $\mathcal{F}_*$ and $\mathcal{H}_*$ on $X=[\ell]\setminus e\subseteq\mathcal{F}^{(1)}$ and $Y=\mathcal{H}^{(1)}\setminus\bigcup_{i_j\in e}V_{i_j}$, *i.e.*,

$$\mathcal{F}_* = \mathcal{F}[X] := \{\mathcal{F}^{(1)}\setminus e, \mathcal{F}^{(2)}[X],\dots,\mathcal{F}^{(h)}[X]\}$$
$$\text{and}\quad \mathcal{H}_* = \mathcal{H}[Y] := \left\{\mathcal{H}^{(1)}\setminus\bigcup_{i_j\in e}V_{i_j}, \mathcal{H}^{(2)}[Y],\dots,\mathcal{H}^{(h)}[Y]\right\}.$$

For a partite isomorphic copy $\mathcal{F}_{0,*}$ of $\mathcal{F}_*$ in $\mathcal{H}_*$, let $\mathrm{EXT}(\mathcal{F}_{0,*})$ be the set of all crossing h-tuples $\eta\in\bigcup_{i_j\in e}V_{i_j}$ such that $V(\mathcal{F}_{0,*}^{(1)})\cup\eta$ spans a partite isomorphic copy of $\mathcal{F}_-$ in $\mathcal{H}_-$, which extends $\mathcal{F}_{0,*}$. Since $\mathcal{F}^{(h)}\subseteq\mathcal{K}_{h-1}(\mathcal{F}^{(h-1)})$, e induces a $K_h^{(h-1)}$ in $\mathcal{F}^{(h-1)}$ and hence $\mathrm{EXT}(\mathcal{F}_{0,*})\subseteq\mathcal{K}_h(\mathcal{H}^{(h-1)}[\bigcup_{i_j\in e}V_{i_j}])$. Set

$$\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*}) = \Delta_{h-1}\big((\mathrm{EXT}(\mathcal{F}_{0,*})\big) = \{\eta'\subset\eta:\ |\eta'| = h-1\ \text{ and }\ \eta\in\mathrm{EXT}(\mathcal{F}_{0,*})\}.$$

Clearly, $\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*})\subseteq\mathcal{H}^{(h-1)}[\bigcup_{i_j\in e}V_{i_j}]$ and $\mathcal{K}_h(\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*}))\supseteq\mathrm{EXT}(\mathcal{F}_{0,*})$. Moreover, a moment's thought shows that, in fact, $\mathcal{K}_h(\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*}))=\mathrm{EXT}(\mathcal{F}_{0,*})$[1]. Hence the regularity of $\mathcal{H}$ yields

$$\begin{aligned}\left|\sum_{\mathcal{F}_{-,0}} 1_{\mathcal{H}^{(h)}}\big(\eta(\mathcal{F}_{-,0})\big) - d_h\right| &= \sum_{\mathcal{F}_{*,0}}\left|\sum_{\eta\in\mathrm{EXT}(\mathcal{F}_{0,*})} 1_{\mathcal{H}^{(h)}}\big(\eta(\mathcal{F}_{-,0})\big) - d_h\right|\\ &\leqslant \#\{\mathcal{F}_*\subseteq\mathcal{H}_*\}\times\varepsilon\left|\mathcal{K}_h\left(\mathcal{H}^{(h-1)}\left[\bigcup_{i_j\in e}V_{i_j}\right]\right)\right|\\ &\leqslant m^{\ell-h}\times\varepsilon m^h\leqslant\varepsilon m^\ell. \end{aligned}\tag{2.3}$$

Combining (2.1)–(2.3) and recalling the choice of ε, we infer

$$\begin{aligned}\#\{\mathcal{F}\subseteq\mathcal{H}\} &= d_h\times\left(1\pm\frac{\gamma}{2}\right)\prod_{j=2}^{h-1} d_j^{|\mathcal{F}^{(j)}|}\times d_h^{|\mathcal{F}^{(h)}|-1}\times m^\ell\pm\varepsilon m^\ell\\ &= \left(1\pm\frac{\gamma}{2}\right)\prod_{j=2}^{h} d_j^{|\mathcal{F}^{(j)}|}\times m^\ell\pm\varepsilon m^\ell = (1\pm\gamma)\prod_{j=2}^{h} d_j^{|\mathcal{F}^{(j)}|}\times m^\ell. \qquad\square\end{aligned}$$

[1] Indeed, the existence of a clique $K\in\mathcal{K}(\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*}))\setminus\mathrm{EXT}(\mathcal{F}_{0,*})$ implies that for some disjoint sets $J\subsetneq K$ and $I\subseteq V(\mathcal{F}_{0,*}^{(1)})$, say $J=\{v_{i_1},\dots,v_{i_j}\}$ and $I=\{v_{i_{j+1}},\dots,v_{i_h}\}$, we have $J\cup I\notin\mathcal{H}^{(h)}$, while $\{i_1,\dots,i_h\}\in\mathcal{F}^{(h)}$. On the other hand, for any $(h-1)$-tuple $\widetilde{H}\in\mathcal{Q}^{(h-1)}(\mathcal{F}_{0,*})$, with $\widetilde{H}\supseteq J$ there exists $H\in\mathrm{EXT}(\mathcal{F}_{0,*})$ with $\widetilde{H}\subset H$, yielding a contradiction.

Theorem 2.2 yields the following corollary, Corollary 2.3, which states that 'most' edges of the h-uniform layer of an $(\varepsilon, \boldsymbol{d}, \mathcal{F}^{(h)})$-regular complex belong to the 'right' number of partite isomorphic copies of $\mathcal{F}^{(h)}$.

Corollary 2.3 (dense extension lemma). *For all integers $2 \leqslant h \leqslant \ell$, every h-uniform hypergraph $\mathcal{F}^{(h)}$ on ℓ vertices and all positive constants γ and d_0, there are $\varepsilon_{\mathrm{DEL}} = \varepsilon_{\mathrm{DEL}}(\mathcal{F}^{(h)}, \gamma, d_0) > 0$ and an integer $m_{\mathrm{DEL}} = m_{\mathrm{DEL}}(\mathcal{F}^{(h)}, \gamma, d_0)$ such that, if $\boldsymbol{d} = (d_2, \dots, d_h) \in \mathbb{R}^{h-1}$ satisfying $d_j \geqslant d_0$ for $2 \leqslant j \leqslant h$ and $m \geqslant m_{\mathrm{DEL}}$, and if $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{h}$ is an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d}, \mathcal{F}^{(h)})$-regular (m, ℓ, h)-complex, then*

$$|\mathcal{H}^{(h)}| = |\mathcal{F}^{(h)}| \times (1 \pm \gamma) \prod_{j=2}^{h} d_j^{\binom{h}{j}} \times m^h, \tag{2.4}$$

and for all but at most $\gamma |\mathcal{H}^{(h)}|$ edges $e \in \mathcal{H}^{(h)}$ we have

$$\mathrm{ext}(e; \mathcal{F}^{(h)}) = (1 \pm \gamma) \prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{F}^{(h)})| - \binom{h}{j}} \times m^{\ell - h}. \tag{2.5}$$

Proof. The proof is based on the following useful consequence of the Cauchy–Schwarz inequality.

Fact 2.4. *For every real $\gamma > 0$, there is some $\beta > 0$ such that, if $x_1, \dots, x_N$ are non-negative real numbers which for some $A \in \mathbb{R}$ satisfy*

$$\sum_{i=1}^{N} x_i = (1 \pm \beta) N A \quad \text{and} \quad \sum_{i=1}^{N} x_i^2 = (1 \pm \beta) N A^2,$$

then for all but at most γN indices $i \in [N]$ we have $x_i = (1 \pm \gamma) A$. □

Let an h-uniform hypergraph $\mathcal{F}^{(h)}$ with vertex set $V(\mathcal{F}^{(h)}) = [\ell]$ and positive reals γ and d_0 be given. We have to find appropriate constants $\varepsilon_{\mathrm{DEL}}$ and m_{DEL}.

First, for every edge f in $\mathcal{F}^{(h)}$, let $\mathcal{D}(\mathcal{F}^{(h)}, f)$ be the h-uniform hypergraph on $2\ell - h$ vertices constructed from two copies of $\mathcal{F}^{(h)}$ by identifying corresponding vertices of the edge f. Now let $\beta \leqslant \gamma$ be given by Fact 2.4 applied with γ. We fix promised constants $\varepsilon_{\mathrm{DEL}}$ and m_{DEL} by setting

$$\varepsilon_{\mathrm{DEL}} = \min\Big\{\varepsilon_{\mathrm{DCL}}\big(h, h, \tfrac{\beta}{3}, d_0\big), \varepsilon_{\mathrm{GDCL}}\big(\mathcal{F}^{(h)}, \tfrac{\beta}{3}, d_0\big), \min_{f \in \mathcal{F}^{(h)}} \{\varepsilon_{\mathrm{GDCL}}\big(\mathcal{D}(\mathcal{F}^{(h)}, f), \tfrac{\beta}{3}, d_0\big)\}\Big\},$$

where $\varepsilon_{\mathrm{DCL}}$ and $\varepsilon_{\mathrm{GDCL}}$ are given by Theorem 2.1 and Theorem 2.2, respectively. Similarly, set

$$m_{\mathrm{DEL}} = \max\Big\{m_{\mathrm{DCL}}\big(h, h, \tfrac{\beta}{3}, d_0\big), m_{\mathrm{GDCL}}\big(\mathcal{F}^{(h)}, \tfrac{\beta}{3}, d_0\big), \max_{f \in \mathcal{F}^{(h)}} \{m_{\mathrm{GDCL}}\big(\mathcal{D}(\mathcal{F}^{(h)}, f), \tfrac{\beta}{3}, d_0\big)\}\Big\}.$$

After we have fixed all constants, let $\boldsymbol{\mathcal{H}} = \{\mathcal{H}^{(j)}\}_{j=1}^{h}$ be an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d}, \mathcal{F}^{(h)})$-regular (m, ℓ, h)-complex with vertex partition $V_1 \cup \dots \cup V_h$, $m \geqslant m_{\mathrm{DEL}}$, and $\boldsymbol{d} = (d_2, \dots, d_h)$ satisfying $d_j \geqslant d_0$

for every $j = 2, \ldots, h$. From the choice of $\varepsilon_{\mathrm{DEL}} \leqslant \varepsilon_{\mathrm{DCL}}\big(h, h, \frac{\beta}{3}, d_0\big)$ and since $m \geqslant m_{\mathrm{DEL}} \geqslant m_{\mathrm{DCL}}\big(h, h, \frac{\beta}{3}, d_0\big)$, Theorem 2.1 (applied to the (m, h, h)-complex $\boldsymbol{\mathcal{H}}[\Lambda_h] = \{\mathcal{H}^{(j)}[\Lambda_h]\}_{j=1}^{h}$ for every $\Lambda_h \in [\ell]^h$ that is an edge in $\mathcal{F}^{(h)}$) yields

$$|\mathcal{H}^{(h)}| = |\mathcal{F}^{(h)}| \times \left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{\binom{h}{j}} \times m^h, \tag{2.6}$$

which implies (2.4). Moreover, since $\varepsilon_{\mathrm{DEL}} \leqslant \varepsilon_{\mathrm{GDCL}}(\mathcal{F}^{(h)}, \frac{\beta}{3}, d_0)$ and

$$m \geqslant m_{\mathrm{DEL}} \geqslant m_{\mathrm{GDCL}}(\mathcal{F}^{(h)}, \tfrac{\beta}{3}, d_0),$$

we can apply Theorem 2.2 to estimate the number of partite isomorphic copies of $\mathcal{F}^{(h)}$ in $\mathcal{H}^{(h)}$ by

$$\left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{F}^{(h)})|} \times m^{\ell}. \tag{2.7}$$

Consequently,

$$\begin{aligned}\sum_{e \in \mathcal{H}^{(h)}} \mathrm{ext}(e; \mathcal{F}^{(h)}) &\overset{(2.7)}{=} |\mathcal{F}^{(h)}| \times \left(1 \pm \frac{\beta}{3}\right) \prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{F}^{(h)})|} \times m^{\ell} \\ &\overset{(2.6)}{=} \frac{1 \pm \frac{\beta}{3}}{1 \pm \frac{\beta}{3}} \times |\mathcal{H}^{(h)}| \times \prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{F}^{(h)})| - \binom{h}{j}} \times m^{\ell - h} = (1 \pm \beta)|\mathcal{H}^{(h)}|A,\end{aligned} \tag{2.8}$$

for

$$A = \prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{F}^{(h)})| - \binom{h}{j}} \times m^{\ell - h}. \tag{2.9}$$

In view of (2.8) and Fact 2.4 it is only left to verify

$$\sum_{e \in \mathcal{H}^{(h)}} \big(\mathrm{ext}(e; \mathcal{F}^{(h)})\big)^2 = (1 \pm \beta)|\mathcal{H}^{(h)}|A^2 \tag{2.10}$$

to prove Corollary 2.3. To this end, let Λ_h be an edge in $\mathcal{F}^{(h)}$. Consider the complex $\boldsymbol{\mathcal{DC}}(\boldsymbol{\mathcal{H}}, \Lambda_h)$, which we obtain by taking two copies $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ of $\boldsymbol{\mathcal{H}}$ and identifying those vertices with its copy which belongs to a vertex class indexed by some $\lambda \in \Lambda_h$.

More explicitly, for $1 \leqslant i \leqslant \ell$ let $V_i = \{v_{1,i}, \ldots, v_{m,i}\}$ be the vertex classes of $\boldsymbol{\mathcal{H}}$. Suppose $W_i = \{w_{i,1}, \ldots, w_{i,m}\}$ and $U_i = \{u_{i,1}, \ldots, u_{i,m}\}$ are the vertex classes of the copies $\boldsymbol{\mathcal{H}}_1 = \{\mathcal{H}_1^{(j)}\}_{j=1}^{h}$ and $\boldsymbol{\mathcal{H}}_2 = \{\mathcal{H}_2^{(j)}\}_{j=1}^{h}$ of $\boldsymbol{\mathcal{H}}$ so that $w_{i,r} \mapsto v_{i,r}$ (respectively, $u_{i,r} \mapsto v_{i,r}$) for every $1 \leqslant i \leqslant \ell$ and $1 \leqslant r \leqslant m$ is an hypergraph isomorphism between $\mathcal{H}_1^{(j)}$ (resp. $\mathcal{H}_2^{(j)}$) and $\mathcal{H}^{(j)}$ for every $j = 2, \ldots, h$. Then, $\boldsymbol{\mathcal{DC}}(\boldsymbol{\mathcal{H}}, \Lambda_h)$ is the complex which we obtain from $\boldsymbol{\mathcal{H}}_1$ and $\boldsymbol{\mathcal{H}}_2$ by identifying $w_{\lambda,r}$ with $u_{\lambda,r}$ for every $\lambda \in \Lambda_h$ and $1 \leqslant r \leqslant m$.

It follows from the assumptions on $\boldsymbol{\mathcal{H}}$ that, for every edge $\Lambda_h \in \mathcal{F}^{(h)}$, the complex $\boldsymbol{\mathcal{DC}}(\boldsymbol{\mathcal{H}}, \Lambda_h)$ is an $(\varepsilon_{\mathrm{DEL}}, \boldsymbol{d}, \mathcal{D}(\mathcal{F}^{(h)}, \Lambda_h))$-regular $(m, 2\ell - h, h)$-complex. Consequently, the earlier choice of $\varepsilon_{\mathrm{DEL}}$ and m_{DEL} allows us to apply Theorem 2.2 to $\boldsymbol{\mathcal{DC}}(\boldsymbol{\mathcal{H}}, \Lambda_h)$ to estimate the number of partite isomorphic copies of $\mathcal{D}(\mathcal{F}^{(h)},\Lambda_h)$ in $\mathcal{DC}(\mathcal{H},\Lambda_h)$ by

$$\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h} d_j^{|\Delta_j(\mathcal{D}(\mathcal{F}^{(h)},\Lambda_h))|}\times m^{2\ell-h}. \tag{2.11}$$

On the other hand, the number of partite isomorphic copies of $\mathcal{D}(\mathcal{F}^{(h)},\Lambda_h)$ in $\mathcal{DC}(\mathcal{H},\Lambda_h)$ coincides with $\sum\{(\mathrm{ext}(e;\mathcal{F}^{(h)}))^2\colon\ e\in\mathcal{H}^{(h)}[\Lambda_h]\}$. Therefore, since $|\Delta_j(\mathcal{D}(\mathcal{F}^{(h)},\Lambda_h))| = 2|\Delta_j(\mathcal{F}^{(h)})|-\binom{h}{j}$ for every $j=2,\dots,h$ we have

$$\sum_{e\in\mathcal{H}^{(h)}[\Lambda_h]}\big(\mathrm{ext}(e;\mathcal{F}^{(h)})\big)^2=\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h} d_j^{2|\Delta_j(\mathcal{F}^{(h)})|-\binom{h}{j}}\times m^{2\ell-h}.$$

Repeating the same argument for every edge $\Lambda_h\in\mathcal{F}^{(h)}$ yields

$$\sum_{e\in\mathcal{H}^{(h)}}\big(\mathrm{ext}(e;\mathcal{F}^{(h)})\big)^2=|\mathcal{F}^{(h)}|\times\left(1\pm\frac{\beta}{3}\right)\prod_{j=2}^{h} d_j^{2|\Delta_j(\mathcal{F}^{(h)})|-\binom{h}{j}}\times m^{2\ell-h}.$$

Hence, in view of (2.9) and (2.6) we have

$$\sum_{e\in\mathcal{H}^{(h)}}\big(\mathrm{ext}(e;\mathcal{F}^{(h)})\big)^2=\frac{1\pm\frac{\beta}{3}}{1\pm\frac{\beta}{3}}\times|\mathcal{H}^{(h)}|\times A^2=(1\pm\beta)|\mathcal{H}^{(h)}|A^2,$$

which gives (2.10) and concludes the proof of Corollary 2.3. □

3. Proofs of Theorem 1.2

The proof of Theorem 1.2 will be a consequence of the results from Section 2, *i.e.*, the dense counting lemma (Theorem 2.1) and dense extension lemma (Corollary 2.3).

Proof of Theorem 1.2. Given integers $\ell\geqslant k\geqslant 2$ and positive constants γ and d_k, set

$$v=\frac{d_k\gamma}{16\binom{\ell}{k}}. \tag{3.1}$$

After fixing v the constant d_0 is displayed and we set

$$\gamma_{\mathrm{DEL}}=\frac{\gamma}{8\binom{\ell}{k}}\times\min\{d_0,d_k\}^{2^\ell}, \tag{3.2}$$

and then for $h=k$ and $\mathcal{F}^{(k)}=K_\ell^{(k)}$ Corollary 2.3 yields positive constants

$$\varepsilon_{\mathrm{DEL}}=\varepsilon_{\mathrm{DEL}}(K_\ell^{(k)},\gamma_{\mathrm{DEL}},\min\{d_0,d_k\})\quad\text{and}\quad m_{\mathrm{DEL}}=m_{\mathrm{DEL}}(K_\ell^{(k)},\gamma_{\mathrm{DEL}},\min\{d_0,d_k\}). \tag{3.3}$$

We finally set $\varepsilon=\min\{\varepsilon_{\mathrm{DEL}},\frac{d_k}{2}\}$ and $m_0=m_{\mathrm{DEL}}$.

Now let $\mathcal{R}=\{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$, $\mathcal{G}^{(k)}$, and $\mathcal{H}^{(k)}$ satisfying assumptions (i)–(iii) of Theorem 1.2 be given. Hence

$$\{\mathcal{R}^{(j)}\}_{j=1}^{(k-1)}\cup\{\mathcal{G}^{(k)}\}$$

is an $(\varepsilon_{\mathrm{DEL}},\boldsymbol{d})$-regular (m,ℓ,k)-complex with $\boldsymbol{d}=(d_2,\dots,d_k)$ and $d_j\geqslant\min\{d_0,d_k\}$ for $j=1,\dots,k$. Observe that the choice of γ_{DEL} in (3.2) yields

$$\gamma_{\mathrm{DEL}}\leqslant\frac{\gamma}{8\binom{\ell}{k}}\prod_{j=2}^{k}d_j^{\binom{\ell}{j}}\leqslant\frac{\gamma}{8\binom{\ell}{k}}\prod_{j=2}^{k}d_j^{\binom{\ell}{j}-\binom{k}{j}}. \tag{3.4}$$

By Definition 7 we may view $\{\mathcal{R}^{(j)}\}_{j=1}^{k-1}\cup\{\mathcal{G}^{(k)}\}$ as an $(\varepsilon_{\mathrm{DEL}},\boldsymbol{d},K_\ell^{(k)})$-regular complex. By the choice of constants in (3.3), we therefore can apply the dense extension lemma, Corollary 2.3, to $\mathcal{G}^{(k)}$ and infer that

$$|\mathcal{G}^{(k)}|=\binom{\ell}{k}\times(1\pm\gamma_{\mathrm{DEL}})\prod_{j=2}^{k}d_j^{\binom{k}{j}}\times m^k, \tag{3.5}$$

and, more importantly, that all but $\gamma_{\mathrm{DEL}}|\mathcal{G}^{(k)}|$ edges $e\in\mathcal{G}^{(k)}$ obey

$$\mathrm{ext}_{\mathcal{G}^{(k)}}\big(e,K_\ell^{(k)}\big)=(1\pm\gamma_{\mathrm{DEL}})\prod_{j=2}^{k}d_j^{\binom{\ell}{j}-\binom{k}{j}}\times m^{\ell-k}. \tag{3.6}$$

In view of the last assertion let $\mathcal{X}\subseteq\mathcal{G}^{(k)}$ be the set of exceptional edges in $\mathcal{G}^{(k)}$. Consequently,

$$|\mathcal{X}|\leqslant\gamma_{\mathrm{DEL}}|\mathcal{G}^{(k)}|, \tag{3.7}$$

and we infer

$$\begin{aligned}
|\mathcal{K}_\ell(\mathcal{G}^{(k)})|&=\frac{1}{\binom{\ell}{k}}\sum_{e\in\mathcal{G}^{(k)}}\mathrm{ext}_{\mathcal{G}^{(k)}}\big(e,K_\ell^{(k)}\big)\geqslant\frac{1}{\binom{\ell}{k}}\sum_{e\in\mathcal{G}^{(k)}\setminus\mathcal{X}}\mathrm{ext}_{\mathcal{G}^{(k)}}\big(e,K_\ell^{(k)}\big)\\
&\overset{(3.6)}{\geqslant}\frac{1}{\binom{\ell}{k}}|\mathcal{G}^{(k)}\setminus\mathcal{X}|\times(1-\gamma_{\mathrm{DEL}})\prod_{j=2}^{k}d_j^{\binom{\ell}{j}-\binom{k}{j}}\times m^{\ell-k}\\
&\overset{(3.7)}{\geqslant}\frac{1}{\binom{\ell}{k}}(1-\gamma_{\mathrm{DEL}})|\mathcal{G}^{(k)}|\times(1-\gamma_{\mathrm{DEL}})\prod_{j=2}^{k}d_j^{\binom{\ell}{j}-\binom{k}{j}}\times m^{\ell-k}\\
&\overset{(3.5)}{\geqslant}(1-\gamma_{\mathrm{DEL}})^3\prod_{j=2}^{k}d_j^{\binom{\ell}{j}}\times m^\ell\geqslant\left(1-\frac{\gamma}{2}\right)\prod_{j=2}^{k}d_j^{\binom{\ell}{j}}\times m^\ell,
\end{aligned} \tag{3.8}$$

where we used $\gamma_{\mathrm{DEL}}\leqslant\gamma/6$ in the last inequality. We also note that (3.7) and (3.4) imply

$$|\mathcal{X}|\leqslant\frac{\gamma}{8\binom{\ell}{k}}\prod_{j=2}^{k}d_j^{\binom{\ell}{j}-\binom{k}{j}}\times|\mathcal{G}^{(k)}|. \tag{3.9}$$

Having estimated the number of cliques in $\mathcal{G}^{(k)}$ we are going to bound the corresponding quantity in $\mathcal{H}^{(k)}$. First observe that

$$|\mathcal{K}_\ell(\mathcal{H}^{(k)})|\geqslant|\mathcal{K}_\ell(\mathcal{H}^{(k)}\cap\mathcal{G}^{(k)})|\geqslant|\mathcal{K}_\ell(\mathcal{G}^{(k)})|-\sum_{e\in\mathcal{G}^{(k)}\setminus\mathcal{H}^{(k)}}\mathrm{ext}_{\mathcal{G}^{(k)}}\big(e,K_\ell^{(k)}\big). \tag{3.10}$$

Since the first term of the last estimate has already been estimated (*cf.* (3.8)), we will now focus on the second. Since $\mathcal{G}^{(k)}$ and $\mathcal{H}^{(k)}$ are ν-close by assumption (iii) of Theorem 1.2, we have

$$|\mathcal{G}^{(k)} \setminus \mathcal{H}^{(k)}| \leqslant \nu|\mathcal{K}_k(\mathcal{R}^{(k-1)})| \leqslant \frac{\nu|\mathcal{G}^{(k)}|}{d_k - \varepsilon} \leqslant \frac{2\nu}{d_k}|\mathcal{G}^{(k)}|, \tag{3.11}$$

where we appealed to the (ε, d_k)-regularity of $\mathcal{G}^{(k)}$ in the second inequality and $\varepsilon \leqslant d_k/2$ in the last one. Consequently,

$$\begin{aligned}
&\sum_{e\in\mathcal{G}^{(k)}\setminus\mathcal{H}^{(k)}} \operatorname{ext}_{\mathcal{G}^{(k)}}\big(e, K_\ell^{(k)}\big) \\
&\quad\overset{(3.6)}{\leqslant} |(\mathcal{G}^{(k)} \setminus \mathcal{H}^{(k)}) \setminus \mathcal{X}|(1+\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} + |\mathcal{X}|m^{\ell-k} \\
&\quad\overset{(3.11)}{\leqslant} \frac{2\nu}{d_k}|\mathcal{G}^{(k)}|(1+\gamma_{\mathrm{DEL}}) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} + |\mathcal{X}|m^{\ell-k} \\
&\quad\overset{(3.9)}{\leqslant} \left(\frac{2\nu}{d_k}(1+\gamma_{\mathrm{DEL}}) + \frac{\gamma}{8\binom{\ell}{k}}\right)|\mathcal{G}^{(k)}| \prod_{j=2}^{k} d_j^{\binom{\ell}{j}-\binom{k}{j}} \times m^{\ell-k} \\
&\quad\overset{(3.5)}{\leqslant} \frac{\gamma}{2} \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^{\ell},
\end{aligned} \tag{3.12}$$

where we also used $\gamma_{\mathrm{DEL}} < 1$ and (3.1) in the last step. Then, (3.8) and (3.12) combined with (3.10), yields

$$|\mathcal{K}_\ell(\mathcal{H}^{(k)})| \geqslant (1-\gamma) \prod_{j=2}^{k} d_j^{\binom{\ell}{j}} \times m^{\ell},$$

which concludes the proof of Theorem 1.2. □

4. Proofs of Theorem 1.3

In this section we deduce Theorem 1.3 from Theorem 1.2. Theorem 1.3 gives a lower bound on the number of cliques in a (δ_k, d_k, r)-regular hypergraph $\mathcal{H}^{(k)}$. In order to apply Theorem 1.2 we have to find an ε-regular $\mathcal{G}^{(k)}$, which is ν-close to $\mathcal{H}^{(k)}$ (*cf.* Definition 6). Such a regular approximation will be provided by the following lemma, which is a simplified version of Lemma 5.1 from [11] (where $\mathcal{F}^{(k)} = \mathcal{K}_k(\mathcal{R}^{(k-1)})$).

Lemma 4.1. *For all positive reals ν and ε, and every vector $\boldsymbol{d} = (d_2, \dots, d_{k-1})$ satisfying $1/d_i \in \mathbb{N}$ for $2 \leqslant i \leqslant k-1$, there exist a positive real $\delta_{4.1}$ and integers $t_{4.1}$ and $m_{4.1}$ such that the following holds. Suppose*

(a) *$m \geqslant m_{4.1}$ and $(t_{4.1})!$ divides m,*
(b) *$\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$ is a $(\delta_{4.1}, \boldsymbol{d})$-regular $(m, k, k-1)$-complex, and*
(c) *$\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ is $(\nu/12, *, t_{4.1}^{2^k})$-regular w.r.t. $\mathcal{R}^{(k-1)}$.*

Then there exists a k-uniform hypergraph $\mathcal{G}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$ such that the following holds:

(i) $\mathcal{G}^{(k)}$ *is* $(\varepsilon, d(\mathcal{H}^{(k)} \mid \mathcal{R}^{(k-1)}))$*-regular w.r.t.* $\mathcal{R}^{(k-1)}$*, and*
(ii) $|\mathcal{G}^{(k)} \triangle \mathcal{H}^{(k)}| \leqslant \nu |\mathcal{K}_k(\mathcal{R}^{(k-1)})|$.

Proof of Theorem 1.3. We will apply Lemma 4.1 to find a 'very regular' hypergraph $\mathcal{G}^{(k)}$ which is ν-close to $\mathcal{H}^{(k)}$. We then apply Theorem 1.2, which in such an environment ensures many ℓ-cliques in $\mathcal{H}^{(k)}$.

Let $\ell \geqslant k \geqslant 2$ be integers and let γ and d_k be positive reals, given by Theorem 1.3. We first have to fix δ_k: For this let

$$\nu_{1.2} = \nu\big(\mathrm{Thm}\,1.2\,(\ell, k, \tfrac{\gamma}{2}, d_k)\big), \tag{4.1}$$

be given by Theorem 1.2. We set δ_k:

$$\delta_k = \frac{\nu_{1.2}}{24}. \tag{4.2}$$

After displaying δ_k, due to the quantification of Theorem 1.3, we get $d_{k-1}, \dots, d_2 > 0$ satisfying $\frac{1}{d_i} \in \mathbb{N}$ for $i = 2, \dots, k-1$ and have to fix constants δ, r, and m_0. For that we first use Theorem 1.2, which gives

$$\begin{aligned} \varepsilon_{1.2} &= \varepsilon\big(\mathrm{Thm}\,1.2\,(\ell, k, \tfrac{\gamma}{2}, d_0 = \min\{d_2, \dots, d_{k-1}, d_k\})\big), \\ m_{1.2} &= m_0\big(\mathrm{Thm}\,1.2\,(\ell, k, \tfrac{\gamma}{2}, d_0 = \min\{d_2, \dots, d_{k-1}, d_k\})\big). \end{aligned} \tag{4.3}$$

As mentioned earlier, we intend to apply Lemma 4.1. For that we now fix the constants

$$\nu_{4.1} = \nu_{1.2}, \quad \varepsilon_{4.1} = \tfrac{1}{2}\varepsilon_{1.2}, \quad \text{and} \quad \boldsymbol{d}_{4.1} = (d_2, \dots, d_{k-1}) \tag{4.4}$$

to obtain the constants

$$\delta_{4.1}, \quad t_{4.1}, \quad \text{and} \quad m_{4.1}.$$

Finally, we fix δ, r, and m_0 required by Theorem 1.3 to

$$\delta = \min\{\tfrac{1}{2}\varepsilon_{1.2}, \tfrac{1}{2}\delta_{4.1}\}, \quad r = t_{4.1}^{2^k}, \quad \text{and} \tag{4.5}$$

$$m_0 = \max\{m_{1.2} + (t_{4.1})!, m_{4.1} + (t_{4.1})!, \tfrac{2}{\gamma}\ell(t_{4.1})!\}. \tag{4.6}$$

Having fixed all constants, let there be given $m \geqslant m_0$, together with a $(\delta, (d_2, \dots, d_{k-1}))$-regular $(m, \ell, k-1)$-complex $\boldsymbol{\mathcal{R}} = \{\mathcal{R}^{(j)}\}_{j=1}^{k-1}$, and a hypergraph $\mathcal{H}^{(k)} \subseteq \mathcal{K}_k(\mathcal{R}^{(k-1)})$, such that $\mathcal{H}^{(k)}$ is (δ_k, d_k, r)-regular w.r.t. $\mathcal{R}^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in [\ell]^k$.

First we obtain a $(\widetilde{m}, \ell, k-1)$-complex $\widetilde{\boldsymbol{\mathcal{R}}} = \{\widetilde{\mathcal{R}}^{(j)}\}_{j=1}^{k-1}$ and a hypergraph $\widetilde{\mathcal{H}}^{(k)} \subseteq \mathcal{K}_k(\widetilde{\mathcal{R}}^{(k-1)})$ from $\boldsymbol{\mathcal{R}}$ and $\mathcal{H}^{(k)}$, respectively, by removing at most $(t_{4.1})!$ vertices from each vertex class so that

$$(t_{4.1})! \text{ divides } \widetilde{m} \quad \text{and} \quad m - (t_{4.1})! \leqslant \widetilde{m} \leqslant m. \tag{4.7}$$

Since we remove only constantly many vertices, we may assume without loss of generality that $\widetilde{\boldsymbol{\mathcal{R}}}$ is a $(2\delta, (d_2, \dots, d_{k-1}))$-regular complex and $\widetilde{\mathcal{H}}^{(k)}$ is $(2\delta_k, d_k, r)$-regular w.r.t. $\widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in [\ell]^k$ and

$$d\big(\widetilde{\mathcal{H}}^{(k)} \mid \widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]\big) = d\big(\mathcal{H}^{(k)} \mid \mathcal{R}^{(k-1)}[\Lambda_k]\big) \pm o(1) = d_k \pm \varepsilon_{4.1}. \tag{4.8}$$

Now we want to apply Lemma 4.1 $\binom{\ell}{k}$ times for every $\Lambda_k \in [\ell]^k$, with the constants chosen in (4.4), to

$$\widetilde{\boldsymbol{\mathcal{R}}}[\Lambda_k] = \{\widetilde{\mathcal{R}}^{(j)}[\Lambda_k]\}_{j=1}^{k-1} \quad \text{and} \quad \widetilde{\mathcal{H}}^{(k)}_{\Lambda_k} = \widetilde{\mathcal{H}}^{(k)} \cap \mathcal{K}_k\big(\widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]\big).$$

Clearly, $\widetilde{\boldsymbol{\mathcal{R}}}[\Lambda_k]$ and $\widetilde{\mathcal{H}}^{(k)}_{\Lambda_k}$ satisfy the assumptions (a)–(c) of Lemma 4.1. We repeatedly apply Lemma 4.1 for every $\Lambda_k \in [\ell]^k$, and infer that, for each $\Lambda_k \in [\ell]^k$, there exists an

$$\big(\varepsilon_{4.1}, d(\widetilde{\mathcal{H}}^{(k)}_{\Lambda_k} \mid \widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k])\big)\text{-regular hypergraph } \widetilde{\mathcal{G}}^{(k)}_{\Lambda_k}$$

which satisfies

$$|\widetilde{\mathcal{G}}^{(k)}_{\Lambda_k} \triangle \widetilde{\mathcal{H}}^{(k)}| \leqslant \nu_{4.1} |\mathcal{K}_k(\widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k])|.$$

Moreover, since $d(\widetilde{\mathcal{H}}^{(k)}_{\Lambda_k} \mid \widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]) = d(\widetilde{\mathcal{H}}^{(k)} \mid \widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]) = d_k \pm \varepsilon_{4.1}$ for every $\Lambda_k \in [\ell]^k$ (*cf.* (4.8)), setting

$$\widetilde{\mathcal{G}}^{(k)} = \bigcup_{\Lambda_k \in [\ell]^k} \widetilde{\mathcal{G}}^{(k)}_{\Lambda_k}$$

gives rise to a sub-hypergraph of $\mathcal{K}_k(\widetilde{\mathcal{R}}^{(k-1)})$, which is $(2\varepsilon_{4.1}, d_k)$-regular w.r.t. $\widetilde{\mathcal{R}}^{(k-1)}[\Lambda_k]$ for every $\Lambda_k \in [\ell]^k$ and which is $\nu_{4.1}$-close to $\widetilde{\mathcal{H}}^{(k)}$. Since $2\varepsilon_{4.1} = \varepsilon_{1.2}$ and $\nu_{4.1} = \nu_{1.2}$ (*cf.* (4.4)), we can apply Theorem 1.2 to $\widetilde{\boldsymbol{\mathcal{R}}}$, $\widetilde{\mathcal{G}}^{(k)}$, and $\widetilde{\mathcal{H}}^{(k)}$, which yields by the choices in (4.1) and (4.3) that

$$|\mathcal{K}_\ell(\widetilde{\mathcal{H}}^{(k)})| \geqslant \left(1 - \frac{\gamma}{2}\right) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times \widetilde{m}^\ell, \tag{4.9}$$

and, consequently, since $\mathcal{H}^{(k)} \supseteq \widetilde{\mathcal{H}}^{(k)}$ we have

$$\begin{aligned}|\mathcal{K}_\ell(\mathcal{H}^{(k)})| &\overset{(4.9)}{\geqslant} \left(1 - \frac{\gamma}{2}\right) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times \widetilde{m}^\ell \\ &\overset{(4.7)}{\geqslant} \frac{1-\gamma}{1-\frac{\gamma}{2}} \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times (m - (t_{4.1})!)^\ell \overset{(4.6)}{\geqslant} (1-\gamma) \prod_{i=2}^{k} d_i^{\binom{\ell}{i}} \times m^\ell. \qquad \square\end{aligned}$$

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